Classical Model of Quantum Noise with the FDTD Method

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Abstract

Numerical models based on the finite-difference time-domain (FDTD) method have been developed to simulate thermal noise and spontaneous emission. Both types of noise may have effects on optical systems. Though their origin lies in quantum mechanics, macroscopic systems in which the discreteness of light can be ignored make it possible to simulate the noise using classical numbers. The absorbing boundary of a one-dimensional (1D) FDTD grid absorbs all incident fields and thus, can be considered a blackbody. For a blackbody to be in thermal equilibrium with its surroundings it must also radiate back into the system. Therefore, the extreme points of the 1D grid act as sources of thermal radiation penetrating into the grid. This method is applied to the study of a 1D leaky optical cavity in transition from the Markovian to a non-Markovian regime. The appropriate spectral properties are given to the noise and the standard result of the quantum Langevin equation is recovered. In a separate treatment, spontaneous emission, which is important to consider in laser dynamics, is simulated through a 1D FDTD based Maxwell-Bloch system. The coupling between the Maxwell and Bloch equations is achieved via a weakly coupled splitting scheme. This model is based on the c-number stochastic differential equations found through the use of the positive P representation. We validate our method by reproducing previous numerical results of superfluorescence. The gain atoms are initially inverted, so inversion-dependent contributions to the stochastic sources are dominant and thus the only sources of noise considered.

1. Introduction

The finite-difference time-domain (FDTD) method [1] has been extensively used in solving Maxwell's equations for dynamic electromagnetic (EM) fields. The incorporation of auxiliary differential equations, such as the rate equations for atomic populations [2] and the Maxwell-Bloch equations for the density-of-states of atoms [3], has lead to comprehensive studies of light-matter interactions. Although the FDTD method has become a powerful tool in computational electrodynamics, it has been applied mostly to classical or semiclassical problems. The light field in an open cavity experiences quantum fluctuations, however, because of its coupling to external reservoirs. The first part of this paper models this quantum noise for the cavity field as a classical noise and incorporates it into FDTD.

Recently, quantum fluctuations due to the spontaneous emission of atoms were introduced to the Maxwell-Bloch equations [4]. This formalism has the advantage of being able to model gain-guided lasers and unstable resonators but employs the slowly-varying-envelope approximation. This approximation will not always hold when considering systems like chaotic open cavities. An FDTD simulation of microcavity lasers including quantum fluctuations was also done recently [5]. White Gaussian noise was added as a source to the electric field at every grid point. The noise amplitude, however, is strictly only related to the excited state's lifetime. The dephasing time which is much shorter than the excited state's lifetime will induce more noise. Thus, in the second part of this paper we develop a more complete method of incorporating spontaneous emission into FDTD.

One advantage of the FDTD method is the direct time-domain calculation of EM fields without prior knowledge of modes. The effective modal behavior is an *emergent* property that results from temporal evaluation of the fields. We intend to introduce noise to the EM field in a way compatible with the FDTD method, namely, without invoking the modal picture. Our goal is to open a new approach for the study of quantum mechanical aspects of radiation in *dynamic* macroscopic systems with classical electrodynamics simulations. We believe our approach has the potential to permit rigorous theoretical investigations of noise in the area of quantum optics and of open systems such as chaotic open cavities.

2. Numerical Model of Thermal Noise

In the modal picture thermal noise is introduced so that the quantum operator of a leaky cavity mode satisfies the commutation relation. For a laser cavity whose loss only comes from the output coupling, the thermal noise is attributed to the thermal radiation that penetrates the cavity through the coupling [6]. Thus the amount of thermal noise depends on the mode decay rate, which must be known in order to solve the Langevin equation for the field operator.

In FDTD simulations, light escaping from an open system is absorbed by the absorbing boundary layer (ABL) which acts as the external reservoir. Since it absorbs all impinging fields, the ABL can be modeled as a blackbody. To remain in thermal equilibrium with temperature T, the blackbody must radiate into the system. The blackbody radiation from the ABL propagates into the cavity and adds as noise to the cavity field. The amount of noise penetrating the cavity depends on the cavity openness or output coupling.

To simulate blackbody radiation, we surround the 1D grid with a series of noise sources $E_s(t_j)$ next to the grid/ABL interface [7]. These soft sources radiate EM waves into the grid having spectral properties consistent with blackbody radiation. The energy density of the blackbody radiation in 1D is

$$D(\omega, T) = \frac{\hbar}{\pi c}\left(\frac{\omega}{\exp(\hbar\omega/kT) - 1}\right), \tag{1}$$

where $\hbar$ is the reduced Planck constant, k is the Boltzmann constant, and ω is the angular frequency. The temporal correlation function of the noise source should be the Fourier transform of $D(\omega,T)$. Freilikher *et al.* have developed a quick and straightforward way in the context of random surfaces to generate random numbers for $E_s(t_j)$ so that these correlations are satisfied [8]. The end result takes advantage of the fast Fourier transform and is given by

$$E_s(t_j) = \frac{\delta}{\sqrt{\tau_{sim}}} \sum_{l=-M}^{M-1} (M_l + i N_l) D_n^{1/2}(|\omega|, T) e^{i\omega_l t_j}, \tag{2}$$

where $2M$ is the total number of time steps, $\tau_{sim} = 2M\Delta t$ is the total simulation time, and $\omega_l = 2\pi l/\tau_{sim}$. M_l and N_l are independent Gaussian random numbers of zero mean and a variance of one. Their symmetry properties are $M_l = M_{-l}$ and $N_l = -N_{-l}$. We normalize the energy density to get $D_n(|\omega|,T)$ in Eq. (2) so that the integral over ω is 2π. The rms amplitude of the noise field δ is adjusted so that the correct EM energy density in vacuum at steady state is achieved, i.e., the unnormalized energy density $D(|\omega|,T)$. This value is found to be $\delta^2=(1/3\epsilon_0\hbar c)(kT)^2$. Using these parameters we have verified the resulting energy density in vacuum to be correct.

To test this model, the field noise in a dielectric slab of length L and refractive index $n > 1$ is calculated. In a good cavity whose lifetime τ is much longer than the coherence time of thermal radiation τ_c, the average amount of thermal noise in one cavity mode agrees with the solution of the quantum Langevin equation under the Markovian approximation. In addition to recovering the standard results, our simulations use various values and combinations of τ and τ_c to illustrate the transition from the Markovian regime to the non-Markovian regime. It is demonstrated that the buildup of the intracavity noise field depends on the ratio of τ_c to τ. This result is explained qualitatively by the interference effect.

3. Numerical Model of Spontaneous Emission

Ziolkowski *et al.* [3] developed an FDTD algorithm for the Maxwell-Bloch equations. They include phenomenological decay rates due to decoherence ($1/T_2$) and excited state's lifetime ($1/T_1$). To advance the algorithm they use a predictor-corrector scheme. We also include these decay rates, but our method of solving these equations differs. Instead, we adopt a method put forth by Bidégaray, called the weakly coupled splitting method [9]. The main idea is to stagger the electric field E and Bloch vector $\boldsymbol{\rho} = \rho_1\boldsymbol{e}_1 + \rho_2\boldsymbol{e}_2 + \rho_3\boldsymbol{e}_3$ updates in time thereby decoupling the equations and creating a simple leap-frog scheme. We have found this to be quite efficient in our 1D simulations. We also choose to simulate $\rho_{22} = \rho_3 + \rho_{11}$ instead of ρ_3, where ρ_{11} and ρ_{22} are the ground state and excited state population respectively. The validity of our scheme was tested using the self-induced transparency results from Ziolkowski *et al.* [3].

Decoherence and the relaxation of the excited state should introduce fluctuations to the system. These quantum mechanical fluctuations can be added to these equations by following the work of Drummond and Raymer [10]. They derive c-number stochastic equations through the positive-P representation which are equivalent to the original operator equations in the limit of a large number of atoms N. The complex field at x_j is $\alpha_j(t)$ and is associated with the noise term $F_j^{\alpha}(t)$. The atomic polarization at cell n at x_j of N_n atoms is $J_n^{+/-}(t)$ and is associated with the noise term $F_n^J(t)$. The atomic inversion $J_n^z(t)$ is associated with the noise term $F_n^z(t)$. We show the noise terms they derived using their original notation in Eq. (3).

$$\begin{aligned} F_j^{\alpha}(t) &= \xi_j^{\alpha}(t)\sqrt{c\kappa <n>}, \\ F_n^J(t) &= \xi_n^J(t)\sqrt{2ig'\alpha_j J_n^-} + \xi_n^P(t)\sqrt{\gamma_P(2J_n^z+N_n)} + \xi_n^0(t)\sqrt{W_{12}N_n}, \\ F_n^z(t) &= \xi_n^z(t)\sqrt{(1/2T_1)(N_n - 2\sigma^{SS}J_n^z) + ig'(J_n^-\alpha_j^{\dagger} - J_n^+\alpha_j) - (2/N_n)W_{12}J_n^+J_n^-} \\ &\qquad - [\xi_n^0(t)J_n^{(2)} + \xi_n^{0*}(t)J_n^-]\sqrt{W_{12}/N_n}, \end{aligned} \tag{3}$$

where $c\kappa$ is the modal-intensity damping rate due to background absorbers, $<n>$ is the average modal photon number, $g' = gM^{1/2}$, g is the coupling parameter, M is the number of grid cells, $\gamma_P = (1/T_2) - (1/2T_1)$ is the "pure dephasing rate," W_{12} is the incoherent pumping rate, and σ^{SS} is the steady state atomic inversion (-1 in the absence of W_{12}). The noise terms $F(t)$ have been reduced to the $\xi(t)$ terms which are δ-correlated independent Gaussian random numbers. $F_j^{\alpha}(t)$ is the fluctuation induced by the decay of the light field to the absorbing background. The first term in $F_n^J(t)$ is the fluctuation due to coupling of the polarization to the light field. The second term in $F_n^J(t)$ is the inversion-dependent classical noise. The last term in $F_n^J(t)$ is the noise due to the pump. The first term under the square root in $F_n^z(t)$ is the inversion dependent classical noise. The second term under the square root is from the coupling of the polarization to the light field. The third term under the square root and the last term in $F_n^z(t)$ are noise due to the pump. Each of these terms should be incorporated into the FDTD Maxwell-Bloch equations to obtain a more complete description of the laser dynamics.

Before taking the step of including every noise term of Eq. (3) in the Maxwell-Bloch equations, a simpler system is considered to test our model. Maki *et al*. examine the transition from superfluorescence (SF) to amplified spontaneous emission (ASE) [11]. The Fresnel number of the system is kept near unity to ensure the accuracy of a 1D description. Due to the fluorescence and collisional fluctuations being large enough to dominate the system, the nonclassical noise terms in Eq. (3) may be dropped. This leaves the $\xi_n^P(t)[(1/T_2)(2J_n^z + N_n)]^{1/2}$ term for the polarization and the $\xi_n^z(t)[(1/2T_1)(N_n - 2\sigma^{SS}J_n^z)]^{1/2}$ term for the inversion. After some algebra, these inversion-dependent contributions can be cast into the Maxwell-Bloch formalism described in the previous section. Figure 1 illustrates the use of our method to examine the transition from SF to ASE in a manner analogous to Fig. 4 in Ref. [11]. The gain αL is directly proportional to the dephasing time T_2. Fig. 1(a) shows results for large T_2 bringing about strong oscillatory SF. As T_2 decreases, the SF becomes damped as seen in Fig. 1(b). When T_2 is below a critical value given by $(\tau_r\tau_D)^{1/2}$, where τ_r is the lifetime and τ_D the delay time of cooperative emission, there are enough collisions to frustrate the cooperative emission. The critical value for the system under study is 16 ps. The transition from damped SF to ASE is seen from Fig. 1(c) to Fig. 1(d). The lifetimes and delay times of each of the four cases agree well with Ref. [11]. We conclude that we have successfully implemented the desired noise into the Maxwell-Bloch equations via FDTD.

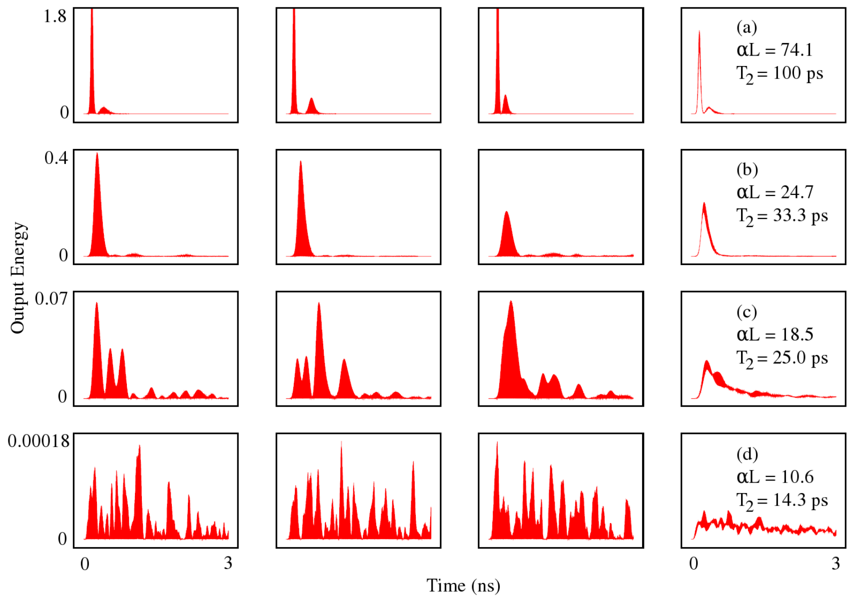

Fig. 1: Output energy of gain region for 3 single realizations (first 3 columns) and an ensemble averaged (using 30 realizations) result. The parameters used are: L = 7 mm, T_1 = 76 ns, $N = 3 \times 10^9$, λ = 629 nm, and Δx = 70 nm. Case (a) shows strong oscillatory SF, (b) damped SF, (c) highly damped SF, and (d) ASE.

4. References

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